Optical-Model Description of Low-Energy Excitations in Nuclear Matter*

C. B. DUKE†
Palmer Physical Laboratory, Princeton University, Princeton, New Jersey
and
General Electric Research Laboratory, Schenectady, New York
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The decay of single-particle excitations in nuclear matter is calculated in the low-density-gas approximation (independent-pair approximation) using a separable nucleon-nucleon interaction. For low-energy excitations this decay is describable in terms of an optical model during a time interval which is longer than 5×10^{-23} sec but shorter than 5×10^{-21} sec. In contrast to earlier calculations, the applicability of the optical model is verified and the mean energy of an excitation is computed prior to the evaluation of the optical-model parameters. Previous calculations of these parameters are shown to lead either to a circumstance in which an optical-model description of the decay is inapplicable or to an excitation whose mean energy is far off the energy shell.

I. INTRODUCTION

IN this paper we study a system which initially consists of nuclear matter in its ground state plus an extra nucleon in an eigenstate of a single-particle Hamiltonian. We utilize a Green's function method[1,2] to evaluate the time dependence of the probability amplitude that this system remains in its initial state. In certain circumstances the time dependence of the actual probability amplitude can be approximated satisfactorily for a limited period of time by that associated with an optical model in which the nuclear medium remains in its ground state while the extra nucleon moves in a complex optical potential. The novel aspects of our analysis, absent from previous treatments of the low-energy optical potential,[3–8] are: (a) our delineation of the requirements which must be satisfied for the optical-model probability amplitude to provide an adequate approximation to the actual probability amplitude, and (b) our explicit verification that an approximate "actual" probability amplitude associated with the composite system may be represented by one associated with the optical model before using the former to calculate the parameters of the optical potential.

In our analysis we utilize a simple two-nucleon interaction which (a) describes low-energy nucleon-nucleon scattering, (b) yields a saturating ground-state energy of nuclear matter, and (c) has previously been employed in a semiphenomenological calculation of the optical potential.[6] In previous computations of the low-energy optical potential,[3–8] the nucleons comprising the nuclear matter have been treated as moving in a self-consistent single-particle potential and the optical potential in which the additional nucleon moves has been obtained by evaluating the forward scattering amplitude due to its (single) scattering from the nuclear-matter nucleons. We show that such a procedure is internally consistent only if the saturation density of nuclear matter and single-particle spectrum of the nuclear-matter nucleons are calculated using the same nucleon-nucleon interactions and essentially identical computational approximations as those used in the evaluation of the optical potential. The use of empirical parameters for the saturation density of nuclear matter and the average binding energy per nucleon, together with a simple two-nucleon interaction which describes low-energy nucleon-nucleon scattering, leads either to the failure of the optical-model description of the single-particle excitations or to the failure of the assumption, implicit in earlier scattering-theory calculations of the optical-model parameters, that the mean energy of the excitation assumes the appropriate noninteracting-particle value (i.e., the mean excitation energy is "on the energy shell"). By the use of consistency criteria in the determination of the single-particle potential the excitations can be constrained to have a mean energy lying on the energy shell. However, these criteria usually are not satisfied.[3,4,6,7]

We also demonstrate that although our internal consistency criteria can be satisfied for arbitrary values of the density of nuclear matter, the single-particle spectrum of the nuclear-matter nucleons must be computed separately for each value of the density. In particular, the single-particle potential which is obtained at the saturation density should not be used in calculations of the optical potential at lower densities. This result implies that Fermi-Thomas estimates of the low-energy optical potential at the surface of a nucleus would have to be performed with more than the customary precision.[4–6]

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† Present address: General Electric Research Laboratory, Schenectady, New York.

[1] V. M. Galitskii and A. B. Migdal, Zh. Eksperim. i Teor. Fiz. **34**, 139 (1958) [English transl.: Soviet Phys.—JETP **7**, 96 (1958)].

[2] A. A. Abrikosov, L. P. Gorkov, and I. E. Dzyaloshinski, *Methods of Quantum Field Theory in Statistical Physics*, translated by R. A. Silverman (Prentice Hall, Inc., Englewood Cliffs, New Jersey, 1963).

[3] K. A. Brueckner, R. J. Eden, and N. C. Francis, Phys. Rev. **100**, 891 (1955); K. A. Brueckner, *ibid.* **103**, 172 (1956).

[4] G. L. Shaw, Ann. Phys. (N. Y.) **8**, 509 (1959).

[5] L. C. Gomes, Phys. Rev. **116**, 1226 (1959).

[6] L. Verlet and J. Gavoret, Nuovo Cimento **10**, 505 (1958).

[7] L. Verlet, Ann. Phys. (Paris) **4**, 644 (1959); Nuovo Cimento **10**, 821 (1958).

[8] J. Sawicki and S. A. Moszkowski, Nucl. Phys. **21**, 456 (1960).

The calculated values of the optical potential are related to those obtained from the phenomenological analysis of nucleon-nucleus (elastic) scattering experiments by assigning (by definition) to the extra nucleon in nuclear matter a wave number determined by the energy the nucleon would have outside nuclear matter.

All of the numerical results presented in this communication were obtained using the Wheeler-Yamaguchi (WY) nucleon-nucleon interaction.[9,10] This interaction is used for two reasons. First, we can solve the independent-pair[11] scattering problem analytically for such a separable interaction. Second, its use permits us to compare our results with those obtained by Verlet and Gavoret[6] in their semiphenomenological investigation of the validity of the classical model.[12] That the WY interaction is an "unrealistic" nucleon-nucleon interaction is evident both from the phase-shift analyses of high-energy nucleon-nucleon scattering[13] and from the meson theory of nuclear forces. Furthermore, the high saturation density of nuclear matter to which it leads invalidates the independent-pair model at the saturation density. This latter fact constitutues the major defect in our numerical calculations. It relegates them to the role of "illustrative" rather than quantitative calculations. Nevertheless, we present an internally consistent perturbation-theoretic treatment of the description of the excitation spectrum in nuclear matter. Although it is unfortunate that previous semiphenomenological calculations proved to be so misleading concerning the sensitivity of the results to the selection of the two-nucleon interaction, the method is sufficiently well-delineated that it now can be applied using more realistic nucleon-nucleon interactions. The modifications of the results presented herein needed for the incorporation of the long-range components of the nucleon-nucleon interaction have been discussed elsewhere.[14]

II. THE GREEN'S FUNCTION DESCRIPTION OF NUCLEAR MATTER

In this part of the paper we initially recapitulate a few well-known[1,2,15] results concerning the Green's function description of large systems. We discuss the relation of these results to the more familiar formulations[16] of the evaluation of the nuclear optical potential. In order to calculate the single-particle Green's function, we adopt the point of view of perturbation theory and focus our attention on the situation in which the range of the two-body interactions is smaller than the mean which $2(k_F/\beta)^3/3\pi^2 \ll 1, \beta^{-1}$ being an average range parameter for the nucleon-nucleon interaction, a suitable internucleon separation in nuclear matter. The possibility of both clustering[17,18] and compressional-mode[19] instabilities in systems of fermions interacting via attractive forces has not been explicitly considered. Two facts suggest that no serious error is incurred by this omission. In addition to other indications that pairing instabilities are relatively unimportant near the observed nuclear density,[17] we find that Cooper poles[18] occur in the independent-pair scattering amplitudes only for densities less than or of the order of one-eighth of the observed nuclear density. However, a more relevant observation is that a zero-energy nucleon outside nuclear matter should give rise to an excitation in nuclear matter with a mean excitation energy above the ground-state energy of about 15 MeV. An approximate calculation of the width of such an excitation should not be sensitive to small errors in the energy spectrum near the ground-state energy.

We carry out the perturbation-theory calculations only for two-body interactions which depend upon scalar combinations of the spin and isotopic spin variables of the interacting nucleons. In particular, most of the results are given only for the WY interaction. As the WY interaction has no hard core, we treat it in both the Hartree-Fock and low-density approximations. After some remarks on the determination of a suitable approximation for the description of short-lived excitations, we finally discuss the calculation of the ground-state energy and the relation of this calculation to the determination of the energy spectrum of low-energy single-particle excitations.

A. Definition of the Boundary Value Problem

In a description of elastic nucleon-nucleus scattering one envisages a system comprised of $A+1$ nucleons in which, both prior to and after the scattering event, one finds an A-nucleon nucleus in its ground state isolated from a nucleon of kinetic energy E_{in}. In this system the A nucleons in the nucleus are bound in a potential

[9] J. A. Wheeler, Phys. Rev. **50**, 643 (1936).
[10] Y. Yamaguchi, Phys. Rev. **95**, 1628 (1954).
[11] H. A. Bethe and J. Goldstone, Proc. Roy. Soc. (London) **A238**, 551 (1957); L. C. Gomes, J. D. Walecka, and V. F. Weisskopf, Ann. Phys. (N. Y.) **3**, 241 (1958); A. Deshalit and V. F. Weisskopf, Ann. Phys. (N. Y.) **5**, 282 (1958); C. B. Duke, Phys. Rev. **136**, B49 (1964), previous paper.
[12] E. Clementel and C. Villi, Nuovo Cimento **2**, 176 (1955); G. C. Morrison, H. Muirhead, and P. A. B. Murdoch, Phil. Mag. **46**, 795 (1955); A. M. Lane and C. F. Wandel, Phys. Rev. **98**, 1524 (1955); S. Hayakawa, M. Kawai, and K. Kikuchi, Progr. Theoret. Phys. (Kyoto) **13**, 415 (1955); K. Harada and N. Oda, *ibid.* **21**, 260 (1959); K. Kikuchi, Nucl. Phys. **12**, 305 (1959); J. Sawicki, Nuovo Cimento **15**, 504 (1960); and T. Terasawa, Nucl. Phys. **39**, 563 (1962).
[13] See, e.g., M. H. MacGregor, M. J. Moravcsik, and H. P. Stapp, Ann. Rev. Nucl. Sci. **10**, 291 (1960).
[14] C. B. Duke, Princeton University, Ph.D. thesis, 1963 (unpublished).
[15] J. M. Luttinger, Phys. Rev. **121**, 942 (1961); A. J. Layzer, *ibid.* **129**, 897 (1963).
[16] F. L. Friedman and V. F. Weisskopf, *Niels Bohr and the Development of Physics*, edited by W. Pauli (Pergamon Press Ltd., London, England, 1955), p. 134; N. C. Francis and K. M. Watson, Phys. Rev. **92**, 291 (1953); V. F. Weisskopf, Rev. Mod. Phys. **29**, 174 (1957).
[17] See, e.g., H. A. Bethe, B. H. Brandow, and A. G. Petschek, Phys. Rev. **129**, 225 (1963); T. Terasawa, Nucl. Phys. **39**, 563 (1962).
[18] See, e.g., R. L. Becker, Phys. Rev. **127**, 1328 (1962).
[19] See, e.g., R. Brout, Phys. Rev. Letters **5**, 193 (1960); L. D. Landau, Zh. Eksperim. i Teor. Fiz. **35**, 97 (1958) [English transl.: Soviet Phys.—JETP **8**, 70 (1959)].

well due to their interactions whereas the impinging and outgoing nucleons are in eigenstates of the kinetic energy operator. If the kinetic energy E_{in} of the incoming nucleon is much greater than the binding energy of those in the nucleus, we can remove the asymmetry between the impinging and bound nucleons by applying the impulse approximation[20] and treating the latter as free. The generation of systematic corrections to this model of the scattering process has been discussed thoroughly.[20] However, our current interest is in values of E_{in} which are comparable to or less than the average binding energy of the bound nucleons. In order to circumvent the asymmetry between the bound and impinging nucleons, we forego an orthodox treatment of the nucleon-nucleus scattering event and analyze an initial-value problem. We consider the time development of the wave function of a composite system which initially is comprised of nuclear matter plus an additional nucleon which is presumed to have penetrated the surface of a "large nucleus" while retaining its free-particle energy despite its entry into a dispersive medium.

The Hamiltonian of the composite system can be written as

$$H=\sum_{\mathbf{k}} E^{(0)}(k)a_{\mathbf{k}}{}^{*}a_{\mathbf{k}}+\tfrac{1}{2}\sum_{\substack{\mathbf{k}_1,\mathbf{k}_2\\ \mathbf{k}_3,\mathbf{k}_4}}\langle \mathbf{k}_1,\mathbf{k}_2|\,V\,|\mathbf{k}_3,\mathbf{k}_4\rangle \times a_{\mathbf{k}_1}{}^{*}a_{\mathbf{k}_2}{}^{*}a_{\mathbf{k}_4}a_{\mathbf{k}_3}, \quad (2.1)$$

$$E^{(0)}(k)=T(k)+V(k), \quad (2.2)$$

$$a_{\mathbf{k}}(t)=\exp(iHt/\hbar)a_{\mathbf{k}}\exp(-iHt/\hbar). \quad (2.3)$$

We restrict our attention to systems which are isotropic so that $E^{(0)}(k)$ depends only on $k=|\mathbf{k}|$.[21] The single-particle kinetic energy is denoted by $T(k)$ and the single-particle potential energy by $V(k)$. The single-particle basis states with respect to which the matrix elements $\langle \mathbf{k}_1,\mathbf{k}_2|\,V\,|\mathbf{k}_3,\mathbf{k}_4\rangle$ of the nucleon-nucleon interaction are defined are given by

$$\phi_{\mathbf{k}}=(\Omega)^{-1/2}\exp(i\mathbf{k}\cdot\mathbf{r})\chi^{m}{}_{1/2}\lambda^{u}{}_{1/2} \quad (2.4)$$

in which $\mathbf{r}$ is the position of the nucleon, m is its spin projection quantum number, and u is its isotopic spin projection quantum number. The latter two variables have been suppressed in our generic "$\mathbf{k}$" notation. The symbol Ω denotes the volume of the large system, and the $a_{\mathbf{k}}(t)$ are the Heisenberg representations of the annihilation operators $a_{\mathbf{k}}$ defined on an antisymmetrized basis of the single-particle states given by (2.4). Using Eqs. (2.1) through (2.4) we replace the original scattering problem by an initial-value problem in nuclear matter. The initial (Schrödinger) state vector of the composite system is schematically indicated by

$$|\chi(t=0)\rangle=a_{\mathbf{k}}{}^{*}(0)\,|\Psi_0\rangle, \quad (2.5a)$$

$$E^{(0)}(k)=E_{\text{in}}, \quad (2.5b)$$

for any selection of the single-particle potential $V(k)$. We have denoted by $|\Psi_0\rangle$ the state vector of the lowest energy eigenstate of the nuclear medium. It is evident that if $V(k)\neq 0$ then both the nucleons originally in the nuclear matter and the extra nucleon all find themselves in the same single-particle potential. We use the definition of energy conservation through the "surface region" of nuclear matter to calculate via equation (2.5b) the value of the wave number of the extra nucleon in terms of its free-particle kinetic energy. When the extra nucleon is inserted it polarizes the medium and thereby influences the latter's wave function. It is convenient in perturbation-theory calculations of $|\chi(t=0)\rangle$ and related quantities to utilize definitions of $E^{(0)}(k)$ which incorporate some of the effects of this polarization. In the Hartree-Fock and low-density approximations, the polarization occurs via the exclusion principle and the forward scattering of the nucleons in the medium from the inserted nucleon. In principle, these effects are accounted for by including $\mathbf{k}$ as an occupied state in the definition of the self-consistent potential $V(k)$. In practice, we usually describe the single-particle potential in the effective-mass approximation and ignore the polarization of the medium.

The probability amplitude that the system with the state vector $|\chi(t=0)\rangle$ at $t=0$ will still be described by the same state vector at a subsequent time $t>0$ is known to be[1,2]

$$P_{\mathbf{k}}(t)=\langle\Psi_0|\,a_{\mathbf{k}}(t)a_{\mathbf{k}}{}^{*}(0)\,|\Psi_0\rangle=-iG(\mathbf{k},t);\quad t>0. \quad (2.6)$$

The quantity $G(\mathbf{k},t)$ is the time-ordered, single-particle Green's function defined by

$$G(\mathbf{k},t)=i\langle\Psi_0|\,T[a_{\mathbf{k}}(t)a_{\mathbf{k}}{}^{*}(0)]\,|\Psi_0\rangle \quad (2.7)$$

in which T denotes the Wick time ordering operator.[1,2] The Fourier transform of $G(\mathbf{k},t)$ is given by

$$G(\mathbf{k},E)=\int_{-\infty}^{\infty}\exp(iEt/\hbar)G(\mathbf{k},t)dt. \quad (2.8)$$

The properties of $G(\mathbf{k},E)$ as a function of the complex variable E for fixed values of $\mathbf{k}$ have been studied extensively.[1,2,14,15] We note only that $G(\mathbf{k},E)$ admits a Lehmann-Källen representation

$$G(\mathbf{k},E)=G^{(+)}(\mathbf{k},E)+G^{(-)}(\mathbf{k},E), \quad (2.9a)$$

$$G^{(+)}(\mathbf{k},E)=\lim_{\Delta\to 0^{+}}\int_0^{\infty}\frac{g_{\mathbf{k}}{}^{(+)}(x)dx}{x+\mu-E-i\Delta}, \quad (2.9b)$$

$$G^{(-)}(\mathbf{k},E)=\lim_{\Delta\to 0}\int_0^{\infty}\frac{g_{\mathbf{k}}{}^{(-)}(x)dx}{\mu-x-E+i\Delta}, \quad (2.9c)$$

in which μ is the separation energy of a nucleon from nuclear matter. At the saturation density, the separation energy is independent of the number of nucleons in the nuclear matter. In the future we will not explicitly indicate the $\Delta\to 0^{+}$ limit. The spectral functions

[20] See, i.e., M. L. Goldberger and K. M. Watson, *Collision Theory* (John Wiley & Sons, Inc., New York, 1964), Chap. 11.

[21] See also, W. Kohn and J. M. Luttinger, Phys. Rev. **118**, 41 (1960); J. M. Luttinger and J. C. Ward, *ibid.* **118**, 1417 (1960).

$g_k^{(+)}(x)$ and $g_k^{(-)}(x)$ satisfy the sum rule

$$\int_0^\infty [g_k^{(+)}(x)+g_k^{(-)}(x)]dx=1. \quad (2.10)$$

The optical model consists of an approximate description of the composite system in which changes in the state of nuclear matter are ignored while the additional nucleon is considered to move in a complex single-particle potential. In such a case $P_k(t)$ would become

$$P_k(t)=c(k)\exp(iE_0(k)t/\hbar-\Gamma(k)t/\hbar), \quad (2.11)$$

which is known[14,22] to be inconsistent with Eqs. (2.9). However, if the spectral function

$$g_k^{(+)}(E-\mu)=(\pi i)^{-1}\,\mathrm{Im}[G(\mathbf{k},E)]; \quad E>\mu \quad (2.12a)$$

is well approximated by the form

$$\mathrm{Im}G(\mathbf{k},E)\approx[\hbar\Gamma(k)c(k)]/[(E_0(k)-E)^2+\Gamma^2(k)] \quad (2.12b)$$

near $E=E_0(k)$, then when $(E_0(k)-\mu)\gg\Gamma(k)$, Eq. (2.11) provides a satisfactory approximation to Eq. (2.6) during the time interval[14,23]

$$(\hbar/\Delta E)\lesssim t\lesssim[\hbar/\Gamma(k)] \quad (2.13)$$

in which ΔE is the energy interval over which the approximation of (2.12a) by (2.12b) is satisfactory. In the event that $\Delta E\gtrsim 2\Gamma(k)$, we say that the system exhibits approximate single-particle motion[14] and that the optical model is valid for scattering events in which the nucleon-nucleus interaction occurs during time intervals specified by (2.13). The depths of the real and imaginary parts of the optical potential are given by

$$V(E_{\mathrm{in}})=E_0(k)-T(k), \quad (2.14a)$$

$$W(E_{\mathrm{in}})=-\Gamma(k), \quad (2.14b)$$

respectively, as functions of the bombarding energy E_{in}.

B. Evaluation of the Green's Function by Perturbation Theory

In this section we present two perturbation-theory calculations of $G(\mathbf{k},E)$: the Hartree-Fock approximation and the low-density approximation. By using perturbation theory the Green's function may be written as

$$G(\mathbf{k},E)=[G_0^{-1}(k,E)-\Sigma(\mathbf{k},E)]^{-1}, \quad (2.15)$$

$$G_0(k,E)=\hbar\left[\frac{1-n^{(0)}(k)}{E^{(0)}(k)-E-i\Delta}+\frac{n^{(0)}(k)}{E^{(0)}(k)-E+i\Delta}\right], \quad (2.16a)$$

$$n^{(0)}(k)=\begin{cases}1; & k=|\mathbf{k}|<k_F\\ 0; & k=|\mathbf{k}|>k_F\end{cases}, \quad (2.16b)$$

$$k_F=[6\pi^2 n/(2s+1)(2i+1)]^{1/3}, \quad (2.16c)$$

in which k_F is the Fermi wave number of the noninteracting Fermi gas used as the zeroth-order description of the composite system; s is the spin of the individual fermions; i is their isotopic spin; n is the particle density of the system; and $E^{(0)}(k)$ is specified by Eq. (2.2). The quantity $\Sigma(\mathbf{k},E)$, referred to as the proper self-energy of a nucleon characterized by the quantum numbers $\mathbf{k}$, is expressed in perturbation theory as an infinite sum of diagrams. Only the final expressions obtained by the use of diagrammatic methods are presented herein because the methods themselves have been described elsewhere.[2,14]

Our analysis has been performed for nucleon-nucleon interactions which depend upon scalar combinations of the spin and isotopic spin variables of a pair of interacting nucleons. In a representation characterized by the spin (S) and isotopic spin (I) of the pair, together with the momenta of the individual fermions, the matrix elements of the nucleon-nucleon interaction become:

$$\langle I',S',\mathbf{k}_1,\mathbf{k}_2|\,V\,|I,S,\mathbf{k}_3,\mathbf{k}_4\rangle_A=\delta_{I,I'}\delta_{S,S'}[\langle\mathbf{k}_1,\mathbf{k}_2|\,V_{IS}\,|\mathbf{k}_3,\mathbf{k}_4\rangle-(-1)^{I+S}\langle\mathbf{k}_2,\mathbf{k}_1|\,V_{IS}\,|\mathbf{k}_3,\mathbf{k}_4\rangle]. \quad (2.17)$$

The subscript A signifies that the matrix element is evaluated with respect to an antisymmetrized two-body basis; $\delta_{I,I'}$ denotes the Kronecker delta; and V_{IS} designates the nucleon-nucleon interaction in the isotopic-spin state I and spin state S. We introduce the definitions

$$\mathbf{P}=\mathbf{k}_3+\mathbf{k}_4, \quad (2.18a)$$

$$\mathbf{P}'=\mathbf{k}_1+\mathbf{k}_2, \quad (2.18b)$$

$$\mathbf{p}=\tfrac{1}{2}(\mathbf{k}_3-\mathbf{k}_4), \quad (2.18c)$$

$$\mathbf{p}'=\tfrac{1}{2}(\mathbf{k}_1-\mathbf{k}_2). \quad (2.18d)$$

Confining our attention to the WY interaction,[9,10] we obtain

$$\langle\mathbf{k}_1,\mathbf{k}_2|\,V_{IS}\,|\mathbf{k}_3,\mathbf{k}_4\rangle=\delta(\mathbf{P}-\mathbf{P}')(-\hbar^2\lambda_{IS}/m)g_{IS}(p')g_{IS}(p), \quad (2.19a)$$

$$g_{IS}(p)=(p^2+\beta_{IS}^2)^{-1}, \quad (2.19b)$$

in which m denotes the average nucleon mass and $\delta(x)$ is the Dirac delta function. From (2.17) and (2.19) we see that only the $(I=0, S=1)$ and $(I=1, S=0)$ WY interactions possess nonvanishing antisymmetrized matrix elements. The nonvanishing matrix elements are augmented by a factor of two because of the antisymmetrization. The values of the parameters in (2.19) obtained from the neutron-proton effective range data[13] are presented in Table I. They adequately describe the neutron-proton total cross sections for relative energies

[22] D. J. Thouless, *The Quantum Theory of Many-Body Systems* (Academic Press Inc., New York, 1961), p. 69.
[23] P. Nozières, *Theory of Interacting Fermi Systems* (W. A. Benjamin, Inc., New York, 1964), p. 70.

TABLE I. Parameters of the Wheeler-Yamaguchi (Refs. 9 and 10) interaction obtained from the neutron-proton effective range data (Ref. 13). The Wheeler-Yamaguchi interaction is specified by Eqs. (2.19) in the text.

Spin S	Isotopic spin I	λ_{IS}	β_{IS}
1	0	0.425 F^{-3}	1.46 F^{-1}
0	1	0.154 F^{-3}	1.18 F^{-1}

below 80 MeV and are used throughout our numerical work.

As the WY interaction possesses no hard core we can evaluate the Hartree-Fock contribution to the proper self-energy. Using an arbitrary scalar nucleon-nucleon interaction, we obtain the result

$$\Sigma^{(\mathrm{H.F.})}(\mathbf{k},m,u,E)=-(4\hbar)^{-1}\sum_{I,S,\mathbf{l}}(2I+1)(2S+1)n^{(0)}(l)$$
$$\times(\langle\mathbf{k},\mathbf{l}|\,V_{IS}\,|\mathbf{k},\mathbf{l}\rangle-(-1)^{S+I}\langle\mathbf{l},\mathbf{k}|\,V_{IS}\,|\mathbf{k},\mathbf{l}\rangle), \quad (2.20)$$

which is independent of the spin and isotopic spin projection quantum numbers, m and u, respectively, of the nucleon with wave vector $\mathbf{k}$. The Hartree-Fock contribution to the proper self-energy is evidently independent both of the value of E and of the direction of $\mathbf{k}$. Therefore, we may incorporate it into the zeroth-order single-particle Green's function by use of the single-particle potential

$$V^{(\mathrm{H.F.})}(k,m,u)=V^{(\mathrm{H.F.})}(k)$$
$$\equiv-\hbar\Sigma^{(\mathrm{H.F.})}(k,m,u,E). \quad (2.21)$$

The use of (2.21) eliminates all diagrams with equal-time self-energy insertions from the perturbation-theory expression for the proper self-energy.[14,24] In the event that the extra nucleon is inserted with $|\mathbf{k}|=k_F$ and the nucleons interact via the WY interaction we obtain

$$V^{(\mathrm{H.F.})}(k)=-(12\pi\hbar^2/m)\sum_{I,S}(\lambda_{IS}/\beta_{IS})$$
$$\times\Big\{\tan^{-1}[(k+k_F)/2\beta_{IS}]+\tan^{-1}[(k_F-k)/2\beta_{IS}]$$
$$-(\beta_{IS}/k)\ln\Big[\frac{(k_F+k)^2+4\beta_{IS}^2}{(k_F-k)^2+4\beta_{IS}^2}\Big]\Big\}. \quad (2.22)$$

If the added nucleon does not lie at the Fermi surface special provision must be made for its contribution to the $\mathbf{l}$ sum in (2.20).

Envisaging nuclear matter to be a low-density gas in which $2(k_F/\beta)^3/3\pi^2\ll1$, β^{-1} being an average range parameter for the nucleon-nucleon interaction, a suitable approximation to the proper self-energy is obtained by selecting only those diagrammatic contributions to it which are linear in the density.[2,14,25] In the absence of external fields (i.e., any single-particle potentials arise solely from the nucleon-nucleon interactions themselves) we obtain the low-density approximation to the proper self-energy:

$$\hbar\Sigma^{(\mathrm{L.D.})}(\mathbf{k},m,u,E)$$
$$=V(k)-(4)^{-1}\sum_{I,S,\mathbf{l}}(2S+1)(2I+1)n^{(0)}(l)$$
$$\times[\langle\mathbf{k},\mathbf{l}|\,T_{IS}(E+E^{(0)}(l))\,|\mathbf{k},\mathbf{l}\rangle$$
$$-(-1)^{S+I}\langle\mathbf{l},\mathbf{k}|\,T_{IS}(E+E^{(0)}(l))\,|\mathbf{k},\mathbf{l}\rangle], \quad (2.23)$$

which, like the Hartree-Fock self-energy, is independent of m, u, and the direction of $\mathbf{k}$. Therefore, we will no longer explicitly indicate these variables in our symbol for the proper self-energy. The quantity $T_{IS}(x)$ is defined by the integral equation

$$\langle\mathbf{k},\mathbf{l}|\,T_{IS}(x)\,|\mathbf{k},\mathbf{l}\rangle=\langle\mathbf{k},\mathbf{l}|\,V_{IS}\,|\mathbf{k},\mathbf{l}\rangle+\sum_{\mathbf{k}',\mathbf{l}'}\frac{\langle\mathbf{k},\mathbf{l}|\,V_{IS}\,|\mathbf{k}',\mathbf{l}'\rangle\cdot(1-n^{(0)}(k'))(1-n^{(0)}(l'))\langle\mathbf{k}',\mathbf{l}'|\,T_{IS}(x)\,|\mathbf{k},\mathbf{l}\rangle}{x-E^{(0)}(k')-E^{(0)}(l')+i\Delta}. \quad (2.24)$$

We require that $E^{(0)}(k)$ be independent of the spin and isotopic spin projection quantum numbers. Such is the case for the kinetic energy and, as we see from (2.20) and (2.21), the Hartree-Fock single-particle potential.

Equation (2.24) is evidently just a restatement of the Brueckner-Bethe-Goldstone equation.[11] For computational convenience we treat the single-particle energies $E^{(0)}(k)$ in the effective mass approximation, i.e.,

$$V(k)=-V_0+\hbar^2k^2/2\sigma, \quad (2.25a)$$
$$E^{(0)}(k)=-V_0+\hbar^2k^2/2m^*, \quad (2.25b)$$
$$m^*=m\sigma/(m+\sigma). \quad (2.25c)$$

We do not consider the extension of Eqs. (2.25) to momentum-dependent values of the effective mass.[5,26] For the single-particle potential given by (2.25), Eq. (2.24) can be solved analytically for separable nucleon-nucleon interactions as discussed in Ref. 27, hereafter referred to as DI. The scattering amplitudes calculated in DI are not directly proportional to the T matrix of Eq. (2.24) because in the analytically continued scattering amplitudes the value of x is a function of the values of $\mathbf{k}$ and $\mathbf{l}$. The solution to (2.24) is obtained from Eq. (13) in DI by (a) requiring $k=k'$ in the

[24] J. Goldstone, Proc. Roy. Soc. (London) **A239**, 267 (1957).

[25] In the case of hard-core two-body interactions β^{-1} is taken to be the zero-energy scattering amplitude. Detailed references to the literature may be found in Ref. 2.

[26] See also, e.g., J. Dabrowski and J. Sawicki, Nucl. Phys. **22**, 318 (1961).

[27] C. B. Duke, Ref. 11.

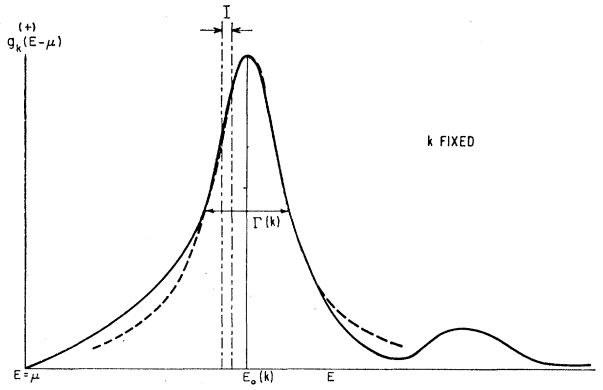

FIG. 1. The solid line is a schematic representation of the spectral function associated with a system which exhibits approximate single-particle motion. The dashed line shows the model Lorentzian spectral function which best approximates the actual spectral function near the latter's maximum. The width $\Gamma(k)$ of the model spectral function and an appropriate energy-averaging interval I are also indicated.

numerator of that formula and giving them both the value $|p|$ refined in (2.18c), and (b) calculating χ in the denominator of (13) in DI by use of Eq. (16) relating k_0 to our energy variable E via Eqs. (17) and (18). The quantity E in Eq. (2.23) is denoted by E_1 in DI. We have not succeeded in performing the last two of the three **I** integrals in (2.23) analytically. We recall from DI that the T matrix exhibits singularities the location of which depends on the total momentum **P** of an interacting pair and on the Fermi wave number, k_F, as well as on the relative energy of the pair. The numerical methods used to perform the singular integrals occurring in (2.23) are discussed in Sec. III.

Before discussing the low-density approximation in more detail we comment on its relation to the optical-model description of low-energy nucleon-nucleus scattering. The fundamental concept on which the optical model is based is that the model describes the "energy-averaged" cross sections observed in low-resolution experiments.[16,28] Therefore, in our initial value problem we expect it to describe those short-lived excitations obtained from a Green's function averaged over an energy interval $I \gtrsim 0.1$ MeV. The energy average of a function like $G^{(+)}(\mathbf{k},E)$ which has singularities only for $\mathrm{Im}(E) \leq 0$ is conveniently obtained using the definition[28,29]:

$$\langle G(\mathbf{k},E_0)\rangle_{\mathrm{av}} \equiv \int_{-\infty}^{\infty} \frac{(I/\pi)G(\mathbf{k},E)dE}{(E-E_0)^2+I^2} = G(\mathbf{k},\, E_0+iI) \quad (2.26)$$

for an average taken about the real energy E_0. If the concept of an optical model provides an adequate description of the composite system then, in accordance with the results of the previous section, we envisage the situation illustrated in Fig. 1. If (a) $I \ll \Gamma(k)$ and (b) $(E_0-\Gamma(k)) \gg \mu$, as shown in the figure, then by the use of (2.26) and (2.9b) we obtain the result

$$\langle G(\mathbf{k},E_0)\rangle_{\mathrm{av}} \cong \int_0^{\infty} \frac{g_{\mathbf{k}}^{(+)}(x)dx}{x+\mu-E_0-iI} \cong G(\mathbf{k},E_0). \quad (2.27)$$

Because we require only the average Green's function for the calculation of an optical potential, the validity of the low-density approximation as a description of the composite system is a sufficient but not necessary condition for the approximation to provide an adequate estimation of the validity and parameters of the optical model. As emphasized by Brown,[28] the necessary conditions for the applicability of (2.23) in describing the decay of an energy-averaged single-particle excitation are (a) the dominance of incoherent two-body scatterings in determining the initial decay mechanism of the excitation, and (b) the requirement that the replenishment of the initial state by processes of higher order in the density (i.e., multiple scattering) create fine structure in the spectral function whose energy width, Γ_{corr}, is considerably less than $\Gamma(k)$. If these conditions are satisfied, then by selecting $\Gamma_{\mathrm{corr}} < I < \Gamma(k)$ we can eliminate via Eq. (2.27) the influence of the higher order terms on the average single-particle Green's function. The extent to which the higher order effects are evident in a particular scattering experiment depends on the energy resolution with which the experiment is performed.[4,30]

Once the nucleon-nucleon interaction has been specified and the saturation density of nuclear matter calculated in the low-density approximation, we justify the use of the low-density approximation by estimating higher order corrections to the proper self-energy. The long-range components of the nucleon-nucleon interaction could cause the excitation of collective modes in nuclear matter to be an important decay mechanism for the single-particle excitation. The relevant contribution to the proper self-energy may be obtained by summing the contributions due to the particle-hole "polarization" diagrams.[14,31] However, for the nonsingular short-range WY interaction we shall consider only those terms in the expansion for the self-energy which are quadratic in the density. There are four types of these terms: (1) those arising from the use of the low-density propagators in the final term of Eq. (2.24); (2) those resulting from particle-hole interactions which occur after the first scattering of the extra nucleon from

[28] G. E. Brown, Rev. Mod. Phys. **31**, 893 (1959). The author is indebted to Professor Brown for several informative discussions concerning the material presented in the remainder of this section.

[29] A. M. Lane and R. G. Thomas, Rev. Mod. Phys. **30**, 257 (1958), Sec. XI.

[30] See also A. K. Kerman, L. R. Rodberg, and J. E. Young, Phys. Rev. Letters **11**, 422 (1963); R. H. Lemmer and C. M. Shakin, Ann. Phys. (N. Y.) **27**, 13 (1964).

[31] The diagrammatic analysis was first given by J. Hubbard, Proc. Roy. Soc. (London) **A240**, 539 (1957) for the high-density electron gas. G. E. Brown, J. A. Evans, and D. J. Thouless, Nucl. Phys. **24**, 1 (1961) have applied similar methods to describe collective vibrational oscillations in closed-shell nuclei.

one of those in the nuclear matter; (3) those arising from the presence of correlations in the ground-state wave function of nuclear matter; and (4) those resulting from triple interactions between the incident nucleon and one in nuclear matter in which one of the interactions is "shielded" by a particle-hole intermediate state. Detailed estimates of these terms are unwarranted for the WY interaction because the high saturation density to which it leads indicates the *prima facie* failure of a low-density expansion. Both Brown[28] and Shaw[4] have performed semiphenomenological estimates of some of the above terms. Their results are consistent with the above picture provided we select $I \sim \Gamma(k)/3$.

C. Determination of the Ground-State Properties of Nuclear Matter

Galitskii and Migdal[1] have shown that the momentum distribution of the particles in a uniform many-fermion system and its ground-state energy are given by

$$n(\mathbf{k}) = \lim_{\Delta \to 0^+} (i/2\pi\hbar) \int_{-\infty}^{\infty} \exp(iE\Delta/\hbar) G(\mathbf{k},E)\, dE, \qquad (2.28)$$

$$E_G = \lim_{\Delta \to 0^+} (i/2\pi\hbar) \sum_{\mathbf{k}} \int_{-\infty}^{\infty} \exp(iE\Delta/\hbar)$$

$$\times \{T(k) + (1/2)[V(k) - \hbar\Sigma(\mathbf{k},E)]\} G(\mathbf{k},E)\, dE, \qquad (2.29)$$

respectively. The single-particle potential, $V(k)$, in (2.29) is caused by the interactions between the fermions and is not due to external fields. If we use an approximation for $\Sigma(\mathbf{k},E)$ in which it depends only on $|\mathbf{k}|$, then we can incorporate all of $\Sigma(\mathbf{k},E)$ into the definition of $V(k)$ and reduce (2.29) to the form

$$E_G = \sum_{\mathbf{k}} n(k)[T(k) + V(k)/2] \qquad (2.30)$$

in which $n(k)$ is defined by (2.28). The most evident such approximation is the Hartree-Fock approximation (2.20) which, for our nonsingular WY interaction, yields

$$E_G = (3n\Omega\hbar^2 k_F^2/10m)[1 - 40\pi \sum_{(I,S)} M_{IS}], \qquad (2.31a)$$

$$M_{IS} = \lambda_{IS}[1 + U_{IS} \tan^{-1}(U_{IS}) - (U_{IS}^{-2} + 3/2) \ln(1 + U_{IS}^2)], \qquad (2.31b)$$

$$U_{IS} = k_F/\beta_{IS}. \qquad (2.31c)$$

The sum over (I,S) in (2.31a) runs over $(I,S) = (1,0)$ and $(0,1)$. The binding energy per nucleon as a function of the Fermi wave number is shown in Fig. 2. The minimum value of E_G/N, $N = n\Omega$, occurs at $k_F = 2.965\,\mathrm{F}^{-1}$ and is given by

$$T(k_F) + V(k_F) = E_G/N = -39.9 \text{ MeV}.$$

The large saturation density and binding energy per nucleon are characteristic of simple s-wave potentials without a hard core.[32] The single-particle potential, $V^{(\mathrm{H.F.})}(k)$, calculated at the saturation density, is shown in Fig. 3 together with two effective-mass approximations to it. The effective-mass approximation

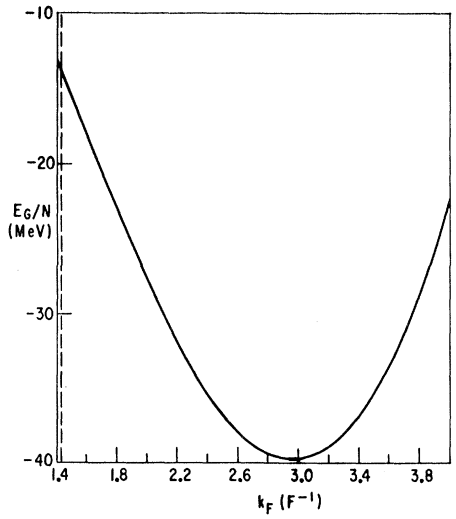

FIG. 2. The solid line represents the binding energy per nucleon, E_G/N, calculated in the Hartree-Fock approximation for the Wheeler-Yamaguchi (Refs. 9, 10) interaction. The dashed line indicates the empirical value of the Fermi wave number (Ref. 37).

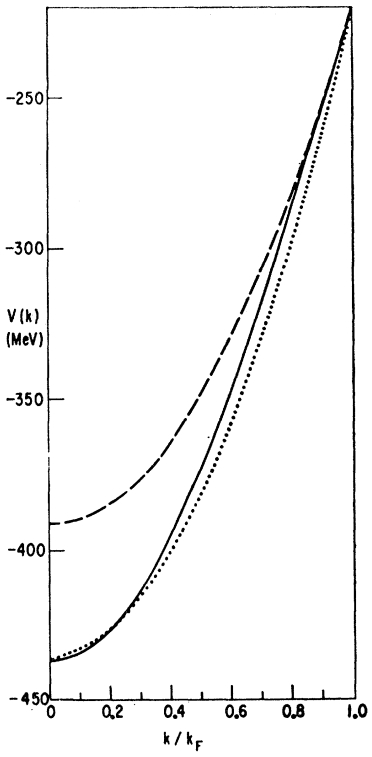

FIG. 3. The solid line represents the single-particle potential, $V^{(\mathrm{H.F.})}(k)$, calculated in the Hartree-Fock approximation for the Wheeler-Yamaguchi[9,10] interaction at the Hartree-Fock value of the saturation density, $k_F = 2.965\ \mathrm{F}^{-1}$. The dashed and dotted lines show two effective-mass approximations to $V^{(\mathrm{H.F.})}(k)$ characterized by $V_0 = 391.7$ MeV, $\sigma = 1.079\,m$ and $V_0 = 435.3$ MeV, $\sigma = 0.858\,m$, respectively.

[32] H. A. Bethe, Phys. Rev. **103**, 1353 (1956).

shown by the dashed line results from selecting $V_0=391.7$ MeV and $\sigma=1.079m$ in (2.25) in order to fit $V^{(\mathrm{H.F.})}(k)$ for $0.9k_F\leq k\leq k_F$. The dotted line shows the best over-all fit to $V^{(\mathrm{H.F.})}(k)$ and is characterized by $V_0=435.3$ MeV and $\sigma=0.858m$. The discrepancy between the two values of σ is a measure of the inadequacy of the effective-mass approximation as a description of the single-particle energy spectrum.

In a perturbation-theory calculation of $\Sigma(k,E)$ we must recognize the special role of the Fermi surface: Any excitation with $k=k_F$ must be stable and possess the separation energy μ defined by[33]

$$E^{(0)}(k_F)-\mu-\hbar\Sigma(k_F,\mu)=0, \tag{2.32}$$

$$\Sigma(k_F,\mu) \text{ is real.} \tag{2.33}$$

The value of μ obtained from (2.32) depends upon the selection of the single-particle potential $V(k)$. If we select $V(k)$ by requiring that the noninteracting-particle term of the Hamiltonian (2.1) yield a description of the system which exhibits the correct separation energy, then we obtain the stronger criterion

$$E^{(0)}(k_F)=\mu, \tag{2.34a}$$

$$\Sigma(k_F,\mu)=0, \tag{2.34b}$$

which is a self-consistency criterion on the choice of $V(k)$. In the Hartree-Fock approximation Eqs. (2.34) are satisfied identically. In general, a sensible definition of a self-consistent energy spectrum for all values of $\mathbf{k}$ is provided by the requirements

$$E_0(k)=E^{(0)}(k), \tag{2.35a}$$

$$\mathrm{Re}[\Sigma(k,E^{(0)}(k))]=0. \tag{2.35b}$$

Equations (2.35) are the analogs of (2.34) for an arbitrary value of $\mathbf{k}$. They define a single-particle spectrum with the property that if the system exhibits approximate single-particle motion in the sense of Sec. A, the mean energy of an excitation lies at the energy predicted by the noninteracting-particle term of (2.1).

In the low-density approximation the exclusion principle requires that $\Sigma^{(\mathrm{L.D.})}(k,E)$ be real for $E<E^{(0)}(k_F)$. Only if (2.34) is used to define μ are the perturbation-theory restrictions due to the exclusion principle treated in a manner which is internally consistent with the more accurate treatment of the dynamics contained in the evaluation of $\Sigma^{(\mathrm{L.D.})}(k,E)$. The application of (2.34) and (2.35) in the low-density approximation is discussed in the Appendix. In this Appendix we also discuss the major sources of the distinction between our approach and the application of Puff's[34] method to uniform nuclear matter.

The final formulas for $n(k)$ and E_G computed in the low-density approximation are given by

$$n^{(\mathrm{L.D.})}(k)=\{1+\hbar\partial[\Sigma^{(\mathrm{L.D.})}(k,E)]/\partial E\}^{-1}{}_{E=E_0(k)}; \tag{2.36}$$

$$E_G=\sum_{\mathbf{k}} n^{(\mathrm{L.D.})}(k)[T(k)+V(k)/2 - (\hbar/2)\Sigma^{(\mathrm{L.D.})}(k,E_0(k))]; \quad E_0(k)<\mu \tag{2.37}$$

$$E_0(k)-E^{(0)}(k)+\hbar\Sigma^{(\mathrm{L.D.})}(k,E_0(k))=0. \tag{2.38}$$

Equation (2.38) defines $E_0(k)$ for an arbitrary selection of $V(k)$ with $\Sigma^{(\mathrm{L.D.})}(k,E)$ being given by Eqs. (2.23) and (2.24). The sum over $\mathbf{k}$ in (2.37) runs only over those values of $\mathbf{k}$ for which $E_0(k)<\mu$. If, in addition to (2.38), we utilize the consistency criterion (2.34), the sum over $\mathbf{k}$ in (2.37) runs over all $|\mathbf{k}|<k_F$. The saturation density is calculated by computing E_G/N for different numerical values of k_F, enforcing the desired consistency criteria for each of the numerical values, and minimizing the resulting E_G/N as a function of k_F. It is well known that the Hartree-Fock approximation yields a minimum value of E_G/N for which the Hugenholtz-Van Hove relation[35] is satisfied, i.e.,

$$E_G/N=(\partial E_G/\partial N)_{\Omega}\equiv\mu=E^{(0)}(k_F). \tag{2.39}$$

However, the low-density approximation is not expected to satisfy (2.39) because (a) $n^{(\mathrm{L.D.})}(k)$ instead of $n^{(0)}(k)$ occurs in Eq. (2.37), and (b) factors of $n^{(0)}(k)$ occur in the second term of (2.24). Our approach can be made analogous to that of Puff[34] via the introduction of coupling between Eqs. (2.23), (2.24), (2.36), and (2.38) by employing $n^{(\mathrm{L.D.})}(k)$ rather than $n^{(0)}(k)$ and $E_0(k)$ in lieu of $E^{(0)}(k)$ in (2.23) [but not in (2.24)].

A systematic low-density calculation of the saturation density and binding energy per nucleon has not been performed. For the nonsingular WY interaction the Hartree-Fock approximation is expected to be reasonably accurate at high densities.[32] The qualitative nature of the anticipated results indicates that more precise computations are unwarranted.

We observe that (a) the Hartree-Fock calculation of E_G/N as a function of k_F yields a curve which is flat near the saturation density, and (b) near the saturation density the values of σ obtained from the Hartree-Fock single-particle spectra are insensitive to changes in the value of k_F. Therefore we can enforce the satisfaction of (2.34) for the calculation of the excitation spectrum by a simple expedient. Using the effective-mass-approximation potential shown by the dotted line in Fig. 3 and computing $\Sigma^{(\mathrm{L.D.})}(k_F,E^{(0)}(k_F))$ from (2.23) and (2.24), we lower the value of k_F from the Hartree-Fock saturation value until (2.34) is satisfied. This procedure yields $k_F=2.940\mathrm{F}^{-1}$ and $\mu=-46.7$ MeV. Although the change in the value of k_F exhibits the opposite sign to that which we would obtain by a systematic application of the low-density approximation, we estimate that μ, V_0, σ, and k_F lie within a few percent of the values which they would assume as a result of their computation in the low-density approximation.

[33] J. M. Luttinger, Phys. Rev. **119**, 1153 (1960).

[34] R. D. Puff, Ann. Phys. (N. Y.) **13**, 317 (1961); D. S. Falk and L. Wilets, Phys. Rev. **124**, 1887 (1961); J. C. Reynolds and R. D. Puff, *ibid.* **130**, 1877 (1963); A. S. Reiner, *ibid.* **133**, B1105 (1964).

[35] N. M. Hugenholtz and L. Van Hove, Physica **24**, 363 (1958).

III. CALCULATIONS OF THE EXCITATION SPECTRA

A. Computational Procedure

The single-particle Green's functions are calculated from (2.15), (2.23), and (2.24) as described in Sec. IIB. Although (2.24) can be solved analytically for the WY interaction, the sum over occupied states in (2.23) must be performed numerically. As the sum over $\mathbf{l}$ exhibits azimuthal symmetry about an axis along $\mathbf{k}$, we use the relation

$$\sum_{\mathbf{l}} n^{(0)}(l) \rightarrow (2\pi)^{-2}\int_0^{k_F} l^2 dl \int_{-1}^{+1} d(\cos\theta_{\mathbf{k},\mathbf{l}}). \quad (3.1)$$

We found in DI that when off-energy-shell propagation is considered, the T matrices exhibit singularities the locations of which are not easily specified. The failure of the independent-pair scattering problem to possess solutions at the values of the relative momentum associated with these singularities indicates that a principal-value boundary condition should be used in the integration in (2.23).[36]

The integrations were performed using a computer program written by the author for the GE 225. In this program provisions are made to automatically locate any singularities of the integrand and provide principal-value boundary conditions about the singular point. In order to save computing time, the integration in the neighborhood of a singularity was adjusted to provide only 0.5% accuracy. The initial numerical grid, prior to the (automatic) detection of rapid variations in the integrand, is specified by

$$\begin{aligned}\cos\theta_{\mathbf{k},\mathbf{l}} &= -1.0(0.142)1.0\,,\\ l &= 0.0k_F(0.2k_F)1.0k_F\,.\end{aligned} \quad (3.2)$$

Sample calculations performed with a more refined grid indicate that the use of (3.2) yields results which are accurate to within 0.5% except for low densities or large values of $|\mathbf{k}|$. In these situations the integrands possess many singularities.

B. Excitation Spectra at the Saturation Density

In Fig. 4 are shown the spectral functions associated with several values of the free-particle energy of the extra nucleon. These spectral functions were computed in the low-density approximation. The single-particle spectrum of the nuclear-matter nucleons was treated in the effective-mass approximation by using the single-particle potential shown by the dotted line in Fig. 3. We see that an optical-model description of the excitations is justified at all of the free-particle energies considered. The parameters of the optical poential are presented in Table II. The sum rule (2.10) is satisfied to within 20% by all of the spectral functions shown in Fig. 4. The spectral functions associated with low values of E_{in} yielded the largest deviations from unity of the integral in (2.10).

[36] A derivation of the validity of this result for the calculation of the ground-state energy is found in Ref. 18.

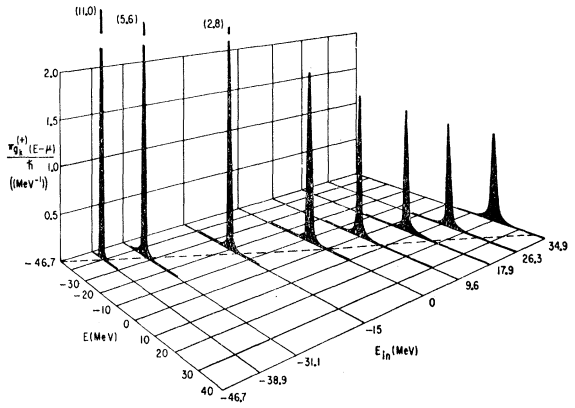

FIG. 4. The solid curves represent spectral functions calculated using the Wheeler-Yamaguchi interaction (Refs. 9, 10), in the low-density-gas approximation from Eqs. (2.1), (2.2), (2.12a), (2.15), (2.23), and (2.24) in the text. The single-particle potential used in the calculation is shown by the dotted line in Fig. 3. The value of k_F was taken to be 2.940 F^{-1}. The dashed line in the figure indicates the energy-shell value of $E=E_{\text{in}}$.

We see in Fig. 4 that, as the energies of the excitations move further above the Fermi energy, they also move below the corresplnding noninteracting particle energies. This "off-energy-shell propagation" of the excitations reflects the inadequacy of the effective-mass approximation for the single-particle energies. This aspect of the effective-mass approximation has long been realized.[5,11,26] In our formalism it can be remedied by the use of more terms in expansion (2.25a) plus the self-consistency requirement (2.35).

From Table II we find that our values of the imaginary part of the optical potential differ from those obtained by numerous other authors[3–8,12] only in that they rise more slowly with increasing values of E_{in}. This feature of the results is a consequence of the large value of the saturation density obtained with the WY interaction. However, almost all calculations based on the independent-pair model or its semiclassical limit yield values of the imaginary part of the optical potential which are equal to within a factor of abour 2.[14] The novel features of our calculation are that (a) we explicitly verify the applicability of the concept of an optical

TABLE II. Parameters of the optical potential associated with the spectral functions shown in Fig. 4.

k/k_F	E_{in} (MeV)	$V(E_{\text{in}})$ (MeV)	$W(E_{\text{in}})$ (MeV)
1.010	−38.9	−223.4	−0.09
1.020	−31.0	−220.4	−0.18
1.040	−15.0	−214.5	−0.36
1.058	−0.4	−209.2	−0.52
1.070	9.6	−205.7	−0.63
1.080	17.9	−202.8	−0.72
1.090	26.3	−200.0	−0.82
1.100	34.9	−197.3	−0.91

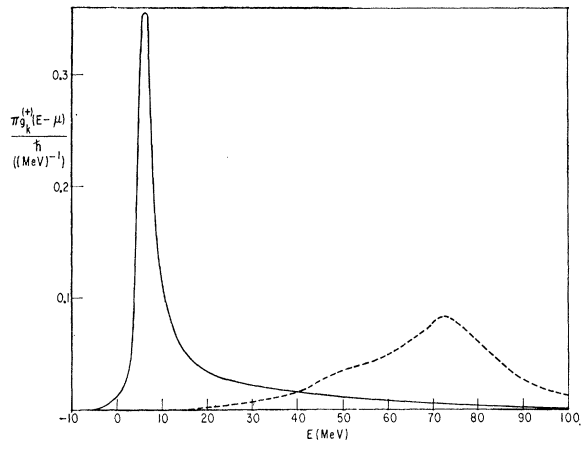

FIG. 5. The solid and dashed lines represent spectral functions associated with $E_{\text{in}}=27.2$ MeV and 79.3 MeV, respectively. The single-particle potential used in the calculations is specified by Eqs. (2.25) in the text with $V_0=47.2$ MeV and $m^*=0.8m$. These values as well as that of $k_F=1.197$ F^{-1}, are identical with the ones used in Ref. 6. The computations were performed using the Wheeler-Yamaguchi (Refs. 9, 10) interaction and Eqs. (2.1), (2.2), (2.12a), (2.15), (2.23), and (2.24) in the text.

model for excitations inside nuclear matter (the only case in which the low-energy independent-pair-model calculations are well defined); and (b) we obtain the actual energy of the excitation without resorting to an *a priori* assumption that it is on the energy shell for some (specified) value of the effective mass. We feel that despite the qualitative nature of the numerical results, the above two accomplishments represent a substantial clarification of the nature of the nuclear-matter estimates of the optical-model parameters. This clarification is particularly interesting because in the next section we demonstrate that for low-energy incident nucleons, the semiphenomenological calculations of VG[6] either do not correspond to a situation in which the concept of an optical model is applicable, or else lead to propagation which is off the energy shell by about 20 MeV in contradiction of their implicit *a priori* assumptions.

C. Failure of Semiphenomenological Calculation of Optical-Model Parameters

We classify as "semiphenomenological" any nuclear-matter calculation of optical-model parameters in which the properties of the nuclear matter are not obtained by using the same nucleon-nucleon interactions and equivalent approximations as those used in the calculation of the optical-model parameters themselves. For energy-shell propagation our calculations are almost identical to those of VG.[6] Therefore, we can illustrate the difficulties inherent in one particular semiphenomenological calculation by reperforming their analysis using our more general method. Following VG, we consider the nucleons in nuclear matter to move in the single-particle potential given by (2.25) for which $m^*=0.8m$ and V_0 is determined by requiring a nucleon with $|\mathbf{k}|=k_F=1.197\text{F}^{-1}$ to have a binding energy $E^{(0)}(k_F)=\mu=-10$ MeV. We verify the equivalence of their calculation and ours by observing that for $m^*=m$ we recover the results given in Table I of Ref. 6 to within a few percent by using the definitions

$$\mathcal{U}_R\equiv V(k)-\hbar\,\mathrm{Re}[\Sigma^{(\text{L.D.})}(k,E^{(0)}(k))], \quad (3.3a)$$

$$\mathcal{U}_I\equiv-\hbar\,\mathrm{Im}[\Sigma^{(\text{L.D.})}(k,E^{0)}(k))]. \quad (3.3b)$$

The linearization factor of[1,14]

$$\{1+\hbar\partial[\mathrm{Re}\Sigma^{(\text{L.D.})}(k,E)]/\partial E\}^{-1}{}_{E=E^{(0)}(k)} \quad (3.4)$$

has been omitted from the definition of $\mathcal{U}_I$ in order to achieve compatibility with the definitions of VG.

The values of $\mathcal{U}_R$ and $\mathcal{U}_I$ calculated by VG correspond to those of V and W, respectively, in Eqs. (2.14) provided that the assumptions

$$E_0(k)=E^{(0)}(k)=\hbar^2k^2/2m^*-V_0, \quad (3.5a)$$

$$\hbar\{\partial[\mathrm{Re}\Sigma^{(\text{L.D.})}(k,E)]/\partial E\}_{E=E_0(k)}\ll 1. \quad (3.5b)$$

are satisfied. We calculated the spectral functions associated with $E_{\text{in}}=-10$ MeV, 27.2 MeV, and 79.3MeV corresponding to the energies in Table I of Ref. 6. The spectral functions associated with $E_{\text{in}}=27.2$ MeV and 79.3 MeV are shown in Fig. 5. For $E_{\text{in}}=-10$ MeV, the spectral function $g_k^{(-)}(x)$ exhibits a delta-function peak whereas $g_k^{(+)}(x)$ is almost invisible on the scale of Fig. 5. Assumption (3.5b) is satisfied only for $E_{\text{in}}=79.3$ MeV. We see from Fig. 5 that assumption (3.5a) can be satisfied only for $E_{\text{in}}>80$ MeV. The satisfaction of (3.5a) for large values of E_{in} is expected because for these values of E_{in} the impulse approximation is valid.[20] For values of E_{in} below 27 MeV, $E_0(k)$ lies more than 20 MeV below $E^{(0)}(k)$. This failure of assumption (3.5a) indicates the occurrence of a serious inconsistency in the energy-shell calculations of optical-model parameters.

The use of a more recent value of the nuclear density,[37] yielding $k_F=1.42\text{F}^{-1}$, and the volume binding energy per nucleon in nuclear matter of -15.8 MeV[38] fails to mitigate the difficulties inherent in the calculation of VG. This result is illustrated in Figs. 6 and 7 in which spectral functions associated with $E_{\text{in}}=27$ MeV and 79 MeV, respectively, are presented. These figures also demonstrate the importance of using "dressed" nucleons in a polarized medium as the noninteracting-particle basis states in a perturbation-theory calculation. If such a set of basis states is not used (i.e., $m^*=m$ in our calculations), then we see from the figures that an optical-model description of the excitation spectrum is appropriate only at energies sufficiently high that we recover the validity of the impulse approximation. This fact suggests the inadequacy of an analysis of surface effects by the use of a local Fermi-

[37] D. G. Ravenhall, Rev. Mod. Phys. **30**, 430 (1958).
[38] A. E. S. Green, Rev. Mod. Phys. **30**, 569 (1958).

Thomas approximation, a technique which fails to describe the mechanism by which the extra nucleon acquires additional kinetic energy as it moves into the dispersive nuclear medium.

Several calculations similar to those of VG have been performed using Serber interactions with Yukawa form factors.[3,4] As these interactions do not lead to a saturation density of nuclear matter, the corresponding calculations of the optical-model parameters suffer from inconsistencies like those found above. Three of the extant calculations of the optical-model parameters[4,5,39] have been performed using two-body interactions for which the nuclear-matter saturation problem has also been analyzed. However, only Reiner,[39] who employs Puff's methods,[34] utilizes in the optical-model calculation those values of the saturation density, binding energy per nucleon, and single-particle energy spectrum which emerge from the ground-state-energy analysis.

The fundamental origin of the inconsistencies in the semiphenomenological calculations resides in their failure to satisfy (2.34). This failure results from the fact that the simple nucleon-nucleon interactions used to describe the low-energy scattering data are too strongly attractive to cause nuclear matter to saturate at the observed density and binding energy per nucleon. The use of these interactions to describe the low-density

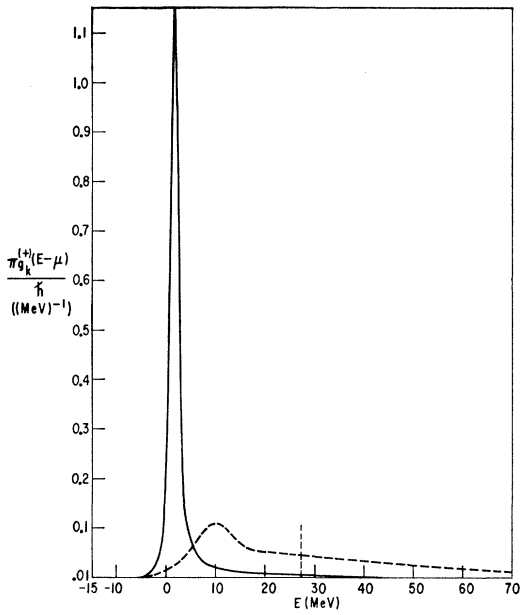

FIG. 6. The solid and dashed lines represent spectral functions calculated using the single-particle potentials specified by Eqs. (2.25) in the text with $V_0=85.7$ MeV, $m^*=0.6m$ and $V_0=57.7$ MeV, $m^*=m$, respectively. Both spectral functions are associated with $E_{\text{in}}=27.1$ MeV and the empirical parameters $\mu=-15.8$ MeV and $k_F=1.42$ F^{-1}. The computations were performed using the Wheeler-Yamaguchi[9,10] interaction and Eqs. (2.1), (2.2), (2.12a), (2.15), (2.23), and (2.24) in the text. The vertical dashed line indicates the energy-shell value $E=E_{\text{in}}$.

[39] A. S. Reiner, Phys. Rev. **133**, B1105 (1964).

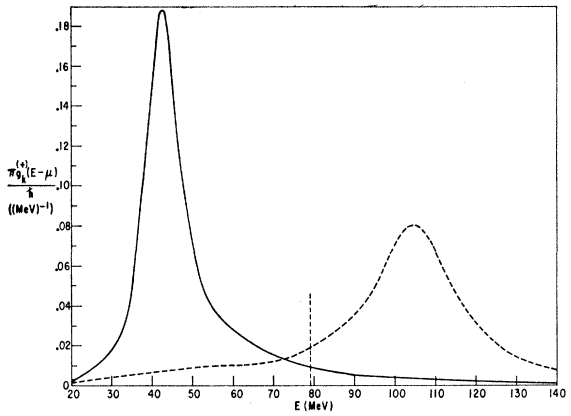

FIG. 7. The solid and dashed lines represent spectral functions calculated using the single-particle potentials specified by Eqs. (2.25) in the test with $V_0=85.7$ MeV, $m^*=0.6m$, and $V_0=57.7$ MeV, $m^*=m$, respectively. Both spectral functions are associated with $E_{\text{in}}=79.0$ MeV and the empirical parameters, $\mu=-15.8$ MeV and $k_F=1.42$ F^{-1}. The computations were performed using the Wheeler-Yamaguchi[9,10] interaction and Eqs. (2.1), (2.2), (2.12a), (2.15), (2.23), and (2.24) in the text. The vertical dashed line indicates the energy-shell value $E=E_{\text{in}}$.

"surface" of a nucleus by means of a local Fermi-Thomas approximation creates not only a more acute form of the inconsistencies encountered above but also leads to the appearance of Cooper poles[18,27] in the T matrices used in computing $g_k^{(+)}(E-\mu)$. The off-energy-shell propagation of the excitations arises from an inadequate treatment of the entry of the extra nucleon into a dispersive medium. The occurrence of Cooper poles indicates the instability of the motion of the extra nucleon with respect to the formation of positive-energy two-body clusters with one of the nucleons in the low-density "surface" of the nucleus.

Although our analysis was motivated by the success of the classical model[12] in describing the imaginary part of the optical potential, we can give no self-evident explanation of this success. We merely observe that in our more complete calculation both the entry of the incident nucleon into a dispersive medium and the exclusion principle play important roles in establishing the validity of an optical-model description of low-energy excitations. Therefore the classical model's reliance solely on the exclusion principle for the reduction of the imaginary part of the optical potential probably represents an oversimplified hypothesis of the mechanism of this reduction even in the special case of the propagation of the extra nucleon through nuclear matter.

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APPENDIX: SELF-CONSISTENT CALCULATION OF THE PROPER SELF-ENERGY IN THE LOW-DENSITY APPROXIMATION

For a system of noninteracting but "dressed" nucleons we find from (2.1) and (2.9) that

$$G^{(+)}(k,E)=\hbar(1-n^{(0)}(k))/(E^{(0)}(k)-E-i\Delta). \quad \text{(A1)}$$

If the system has a specified density, we follow Landau[2] and write (A1) in terms of excitations which correspond to excited states of the "dressed" nucleons.

$$G^{(+)}(k,E)=\hbar(1-n^{(0)}(k))/(\epsilon^{(0)}(k)+\mu-E-i\Delta); \quad \text{(A2)}$$
$$\epsilon(k)=E^{(0)}(k)-\mu.$$

It is well known[2] that (A2) is the zero-temperature analog of a temperature-dependent Green's function such as that used by Puff.[34] The excitations are properly described by use of the grand canonical ensemble because the number of excitations is not a constant of the motion even for a fixed number of nucleons in nuclear matter.

Puff's results are a consequence of using the grand canonical ensemble to describe the nucleons themselves. To recover his results we replace $E^{(0)}(k)$ by $E^{(0)}(k)-\mu$ in the temperature-dependent analog of (A1) and use

$$E^{(0)}(k)=T(k).$$

Puff's simple treatment of the two-body T matrix stems from his observation that $T(k)-\mu>0$ for all values of k. This simplification occurs only when we abandon any self-consistent-field description of the "dressed" nucleons in nuclear matter. In the case described in Sec. C of part II, we would obtain $E^{(0)}(k)-\mu\leq 0$ for all values of $|\mathbf{k}|<k_F$. The application of Puff's methods to this situation leads to a coupled-equation problem like the one discussed in II-C except that a hole-hole propagation term would be added to the analog of (2.24).

A well-known[11] feature of the low-density approximation is the reality of $\Sigma^{(\mathrm{L.D.})}(k,E)$ for $E<E^{(0)}(k_F)$. This result, which follows directly from (2.23) and (2.24), implies that

$$G^{(-)}(k,E)=-2\pi i\hbar\delta(E-E_0(k))/\{1+\hbar\partial[\Sigma^{(\mathrm{L.D.})}(k,E)]/\partial E\}_{E=E_0(k)} \quad \text{(A3)}$$
$$=-2\pi i\hbar n^{(\mathrm{L.D.})}(k)\delta(E-E_0(k)); \quad E_0(k)<E^{(0)}(k_F)$$

in which $n^{(\mathrm{L.D.})}(k)$ and $E_0(k)$ are defined by (2.36) and (2.38), respectively. For $E<E^{(0)}(k_F)$, $\Sigma^{(\mathrm{L.D.})}(k,E)$ has the energy dependence of a R function[40] so that

$$\partial[\Sigma^{(\mathrm{L.D.})}(k,E)]/\partial E>0; \quad n^{(\mathrm{L.D.})}(k)<1. \quad \text{(A4)}$$

We reemphasize[11] that (A3) and (A4), which stem from the reality of $\Sigma^{(\mathrm{L.D.})}(k,E)$ for real values of E, are a direct consequence of the neglect of hole propagation in a low-density system. The validity of (A3) for $E_0(k)=E^{(0)}(k_F)$ is a general feature of perturbation theory.[2,33] Most selections of $V(k)$ (e.g., that obtained by using a semiphenomenological effective-mass approximation with the WY interactions) lead to the consequence that $E_0(k_F)<E^{(0)}(k_F)$. Such a result implies that the influence of the exclusion principle is overestimated in the evaluation of (2.23) and (2.24). In order to mitigate this overestimation while retaining the simplicity inherent in the perturbation theory use of $n^{(0)}(k)$ rather than $n^{(\mathrm{L.D.})}(k)$ in (2.23) and (2.24), we could adopt the following procedure for the calculation of E_G in a system with a specified density.

(1) Determine k_F from the density via (2.16c).

(2) Select for $V(k)$ a form which contains adjustable parameters and evaluate

$$\mu\equiv E^{(0)}(k_F)=T(k_F)+V(k_F). \quad \text{(A5)}$$

(3) Select a value of k and compute $\Sigma^{(\mathrm{L.D.})}(k,E)$ for as many values of E, $E<\mu$, as necessary to determine the solution $E_0(k)$ to Eq. (2.38).

(4) Perform step (3) for a suitable grid of values of k until the function $E_0(k)$, $E_0(k)<\mu$, has been determined. It is likely that $E_0(k_F)<\mu$ and $k_{\max}$ defined by $E_0(k_{\max})=\mu$ satisfies $k_{\max}>k_F$.

(5) Adjust the parameters in $V(k)$ so that $E^{(0)}(k)\cong E_0(k)$ as closely as possible subject to the constraint that $E^{(0)}(k_F)=E_0(k_F)$.

(6) Iterate steps (3) through (5) until the parameters in $V(k)$ do not change appreciably from one iteration to the next.

(7) Compute $E_G(k_F)$ from (2.37) using the self-consistent $V(k)$. Note that $\Sigma^{(\mathrm{L.D.})}(k,E_0(k))$ will be small for $|\mathbf{k}|<k_F$ and zero for $|\mathbf{k}|=k_F$.

The above procedure is evidently similar to that of Brueckner and Gammel.[41] In order to obtain the saturation density and binding energy per nucleon we minimize $E_G(k_F)/N(k_F)$ as a function of k_F. The satisfaction of the Hugenholtz-Van Hove relation[35] and the introduction of coupling between Eqs. (2.15), (2.23), (2.24), (2.36), and (2.38) are discussed in the main text.

[40] E. P. Wigner, Ann. Math. **53**, 36 (1951). The proper self-energy and single-particle Green's function associated with an "infinite" medium are not R functions in general because they are not real for real values of E.

[41] K. A. Brueckner and J. L. Gammel, Phys. Rev. **109**, 1023 (1958).